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Mr. Haldeman shows how a certain quartic curve may be used in order to construct a regular heptagon. Compare a similar article by the same author (*1919*, 390), in which the construction is effected by means of an equilateral hyperbola. Reference may be made also to Note 24 in the department of Problems and Solutions elsewhere in this issue.

I. The Shortest Circular Path from a Point to a Line.

By Arnold Dresden, University of Wisconsin.

Introduction. In connection with the study of a certain type of problem in the calculus of variations, I was led to consider the following question:

"Let there be given a line $x = b$ and a system of circles tangent at the origin to a line $y = mx$ $(m > 0)$ (see figure 1). We take on each of these circles the arc from the origin to the nearer of the two points of intersection with the line $x = b$ and inquire which among the circles of the system furnishes for this arc the minimum value; that is, we inquire which among the circular arcs passing through the origin with slope m will furnish the shortest path from the origin to the line $x = b$."

The intuitive reply to this question was that the shortest circular path is furnished by the circle which cuts the line $x = b$ at right angles. This reply proved to be incorrect. It seemed therefore worth while to publish a discussion of this question which incidentally yielded other points of interest.

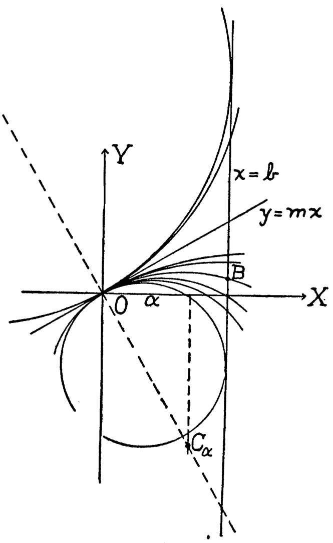

FIG. 1.

1. The circles of the system are given by the equation

$$(x-\alpha)^2+\left(y+\frac{\alpha}{m}\right)^2=\frac{\alpha^2(m^2+1)}{m^2}, \tag{1}$$

the parameter α being the x-coördinate of the center. The points of intersection of this circle with the line $x = b$ are real if $\alpha \geqq b/(1+\mu)$ or $\alpha \leqq b/(1-\mu)$, where

$$\mu=\frac{\sqrt{1+m^2}}{m}=\operatorname{cosec}\phi,$$

$(\phi < \pi/2)$ being the inclination of the line $y = mx$. The length of the arc OB then becomes

$$L(\alpha)=\int_0^b\sqrt{1+y'^2}\,dx=\alpha\mu\int_0^b\frac{dx}{\sqrt{\alpha^2\mu^2-(x-\alpha)^2}}=\alpha\mu\left\{\arcsin\frac{b-\alpha}{\alpha\mu}+\arcsin\frac{1}{\mu}\right\},$$

this function being defined only for values of α greater than $b/(1+\mu)$ or less than $b/(1-\mu)$. The question we raised is that of determining the value of α for which $L(\alpha)$ is a minimum.

2. We introduce as a new variable the angle θ, defined by the relation

$$\theta = \text{arc sin}\frac{b-\alpha}{\alpha\mu} \qquad \text{or} \qquad \alpha = \frac{b}{1+\mu \sin\theta},$$

which transforms[1] the ranges $\alpha \geqq b/(1+\mu)$ and $\alpha \leqq b/(1-\mu)$ into the single continuous range $-(\pi/2) \leqq \theta \leqq (\pi/2)$. It is shown without difficulty that this angle θ represents the supplement of, or the negative, the inclination of the tangent to the circle C_α at its point of intersection with the line $x = b$.

We find then

$$L(\alpha) = L\left(\frac{b}{1+\mu\sin\theta}\right) = L_1(\theta) = \frac{b(\theta+\phi)}{\sin\theta+\sin\phi},$$

and hence

$$L'(\alpha) = L_1'(\theta)\frac{d\theta}{d\alpha} = -\frac{\sin\theta+\sin\phi-(\theta+\phi)\cos\theta}{\sin\phi\cos\theta}.$$

The maximum and minimum values of the function $L_1(\theta)$ will be found among the zeros of the function

$$F(\theta) \equiv \sin\theta + \sin\phi - (\theta+\phi)\cos\theta.$$

While for our purpose we are concerned only with the values of $F(\theta)$ on the range $(-\pi/2, \pi/2)$, we discuss the function in its entirety.

We find $F'(\theta) = (\theta+\phi)\sin\theta$, so that

$$F'(\theta) = 0, \qquad \text{for} \qquad \theta = -\phi, \qquad \text{and for} \qquad \theta = n\pi,$$

while $F'(\theta) > 0$ on the intervals

$$\cdots (-5\pi, -4\pi), (-3\pi, -2\pi), (-\pi, -\phi), (0, \pi), (2\pi, 3\pi), (4\pi, 5\pi) \cdots,$$

and $F'(\theta) < 0$ on the intervals

$$\cdots (-4\pi, -3\pi), (-2\pi, -\pi), (-\phi, 0), (\pi, 2\pi), (3\pi, 4\pi) \cdots.$$

Moreover since

$$F(0) = \sin\phi - \phi < 0, \qquad F(-\phi) = 0, \qquad F(\phi) = 2(\sin\phi - \phi\cos\phi) > 0,$$

we obtain for the graph of $F(\theta)$ a curve as sketched in figure 2. We conclude that there exists a value θ_1 such that $0 < \theta_1 < \phi$ and such that $F(\theta_1) = 0$.

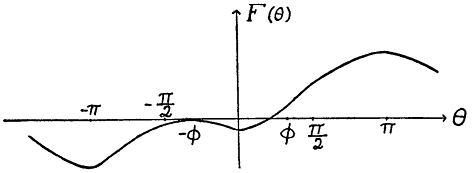

FIG. 2

Corresponding to the values $\theta = -\pi/2, -\phi, 0, \theta_1, \phi, \pi/2$, we have the values $\alpha = b/(1-\mu), \pm\infty, b, \alpha_1, b/2, b/(1+\mu)$, so that we can now obtain the graph of the function $L(\alpha)$ as in figure 3. The values $\alpha = b/(1-\mu)$ and $\alpha = b/(1+\mu)$ yield the circles of the system that are tangent to the line $x = b$ while $\alpha = \pm\infty$ yields the straight line $y = mx$.

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[1] No value of α corresponds to $\theta = -\text{arc sin } 1/\mu$; but

$$\lim \theta = -\text{arc sin } 1/\mu.$$

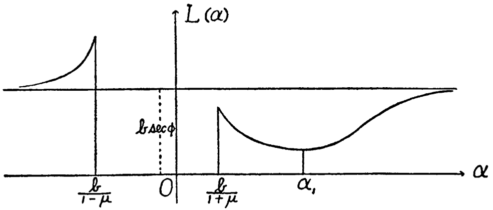

FIG. 3

3. It is readily verified that

$$L\left(\frac{b}{1-\mu}\right) > L(\infty)$$

and

$$L\left(\frac{b}{1-\mu}\right) > L\left(\frac{b}{1+\mu}\right);$$

but that $L[b/(1+\mu)]$ may be greater or less than $L(\infty)$. There is exactly one value of α, viz., α_1, for which $L(\alpha)$ has a minimum; this value α_1 lies between b and $b/2$, so that the shortest circular path from the origin to the line $x = b$ which departs from the origin with slope m lies between the circle which cuts this line perpendicularly and the one which passes through the point $(b, 0)$.

We have seen that $\alpha_1 = b/(1 + \mu \sin \theta_1)$, while θ_1 is the unique root of the equation $\sin \theta + \sin \phi - (\theta + \phi) \cos \theta = 0$ on the interval $(0, \pi/2)$. Hence, θ_1 is determined as a function of ϕ by the relation

$$\frac{\sin \theta_1 + \sin \phi}{\theta_1 + \phi} = \cos \theta_1.$$

From this it follows that $\lim\limits_{\phi \to 0} \theta_1 = 0$; that is, among the circles tangent to the X-axis at the origin, the straight line furnishes the shortest path from the origin to the line $x = b$, which shows that the result obtained above is in accord with known facts.

II. Mathematical Articles in Encyclopedias.

By Norman Anning, University of Michigan.

The efforts of the Association to produce a Mathematical Dictionary that shall be a credit to American scholarship are, in the highest degree, praiseworthy. While that is in progress could not some arrangement be made with publishers who are bringing out important works of reference to have the proofs of mathematical articles read by mathematicians? The editors of the *Encyclopedia Americana* following the policy that articles on technical subjects should be written by experts have, in the main, been singularly happy in their choice of authors. With certain less technical articles they have been less fortunate. For every one who will read through Dr. Webster's excellent treatment of Hydrodynamics there are perhaps a hundred who will consult the *Encyclopedia* to find the "indispensable minimum" of information about the Hyperbola. This is, in part, what the latter will find:[1]

> The length of the transverse axis is said to be a, $\cdots$. The two lines
>
> $$\frac{x}{a} = \pm \frac{y}{b}$$
>
> approach the hyperbola closer than any assigned limit, and are called the assymptotes (sic) of the

---

[1] Vol. 14, p. 601.